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ANALYTIC SYMMETRIES AND THE PROPERTIES OF PHASE TRAJECTORIES*

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This note presents new data on the analyticity of the normalizing transformation of a system in the plane defining the focus, and is a brief supplement to the author's earlier paper /1/.

The key question in proving the analyticity of a group admitted by the equations (in polar coordinates)

$$\rho^{\cdot} = R(\rho, \varphi),\ \varphi^{\cdot} = \Omega(\rho, \varphi) \tag{1}$$

is the establishment of analyticity of solutions of the functional equation

$$\psi(\rho) = \psi(\rho_0)\partial\rho/\partial\rho_0\,|_{\varphi=\omega} \tag{2}$$

where ω is a period of arbitrary multiplicity /1/.

Let us show how the convergence of series

$$\psi(\rho) = \sum_{k=1}^{\infty} A_k \rho^{2h+k} \tag{3}$$

is related to properties of phase trajectories

$$\rho(\rho_0, \varphi) = \sum_{k=1}^{\infty} \beta_k(\varphi)\rho_0^k, \qquad \rho(\rho_0, 0) = \rho_0$$

$$\beta_k(\varphi) = a_0^{(k)}(\varphi) + a_1^{(k)}(\varphi)\varphi + \ldots + a^{(k)}_{k_1}(\varphi)\varphi^{k_1}$$

where $a_\nu^{(k)}(\varphi)$ are trigonometric polynomials of period ω.

Setting in Eq.(2)

$$\rho = \rho(\rho_0, \omega) = \sum_{k=1}^{\infty} b_k\rho_0^k, \qquad \rho^l = \sum_{k=0}^{\infty} B_k^{(l)}\rho_0^{l+k} \tag{4}$$

we obtain the recurrent formulas

$$A_{\sigma-2h+1} = \frac{1}{(\sigma-2h)\omega}\sum_{k=2h+1}^{\sigma} A_{\sigma-k+1}\left[(k+1)b_{k+1} - B_k^{(\sigma-k+2h+1)}\right] \tag{5}$$

Since the quantities $A_\mu\ (\mu \geqslant 1)$ are independent of the multiplicity of period ω, the right-hand sides of formulas (5) depend, in fact, not on coefficients $b_k \equiv \beta_k(\omega)$, $B_k^{(l)}$ but only on their parts b_k', $B_k^{(l)'}$ which are linear in ω. The convergence of series

$$\sum_k a_1^{(k)}(0)\rho_0^k \tag{6}$$

implies the estimates

$$|b_k'| \leqslant \omega\delta^{-k}, \qquad |B_k^{(l)'}| \leqslant l\omega\delta^{-(k+1)}$$

Indeed

$$\rho \equiv \rho(\rho_0, \omega) = \rho_0 + \omega\sum_{k=1}^{\infty} a_1^{(2h+k)}(0)\rho_0^{2h+k} + O(\omega^2) \quad \text{hence} \quad \rho^l = \rho_0^l + l\omega\sum_{k=1}^{\infty} a_1^{(2h+k)}(0)\rho_0^{2h+k+l-1} + O(\omega^2)$$

By definition $B_k^{(l)'}$ is the linear in ω part of coefficient $B_k^{(l)}$ of the expansion of ρ^l in series in the initial ρ_0. Consequently

$$b_k' = B_{k-1}^{(1)'} = a_1^{(k)}(0)\,\omega, \qquad B_k^{(l)'} = l\omega a_1^{(k+1)}(0)$$

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Taking these formulas into account, we obtain from the convergence of series (6) the above estimates, which enables us, using formulas (5), to prove the convergence of series (3). Besides, as will be clear subsequently, it is also possible to obtain exact expressions for the quantities A_k.

Estimates $|b_k| < \delta^{-k}$, $|B_k^{(l)}| < \delta^{-k}$ were used in /1/ at the same stage of the proof. A.D. Briuno had correctly pointed out (but did not properly prove) the inaccuracy of the last of estimates (*). However the above analysis shows that the presence of that inaccurate estimate does not violate the proof. But the content of the theorem proved in /1/ is affected, in so far as there is a direct relation between the convergence of series (4) and (6) and, consequently, the analyticity of the group symmetry of system (1) can be guaranteed only in the case of functions R and Ω for which series (6) is convergent.

Group considerations enable us to supplement somewhat the above investigation.

Let us consider the series

$$\rho^* = \rho_0 + \sum_{k>1} (a_1^{(k)}(0)\,\varphi^* + \ldots + a_{k_1}^{(k)}(0)\,\varphi^{*k_1})\,\rho_0^k \equiv F(\rho_0, \varphi^*) \tag{7}$$

The differential equation $d\rho/d\varphi = R/\Omega$ of phase trajectories admits a discrete group admits of the polar angle $\varphi' = \varphi + m\omega$, where m are integers. On the basis of this it is possible to establish that the mappings $\rho^* = F(\rho_0, \varphi^*)$, considered as transformations of the polar radius, constitute the one-parameter group

$$F(F(\rho_0, \varphi_1^*), \varphi_2^*) = F(\rho_0, \varphi_1^* + \varphi_2^*),\ F(F(\rho_0, \varphi^*), -\varphi^*) = \rho_0,\ F(\rho_0, 0) = \rho_0.$$

Consequently the series $F(\rho_0, \varphi^*)$ satisfies the differential equation

$$\frac{d\rho^*}{d\varphi^*} = \frac{\partial F}{\partial \varphi^*}\Big|_{\varphi^*=0,\ \rho_0=\rho^*} = \sum_k a_1^{(k)}(0)\,\rho^{*k} \equiv f(\rho^*) \tag{8}$$

On the other hand, $\psi(\rho^*)\partial/\partial\rho^*$ in conformity with Eq.(2) is the operator of the group that transforms the set of mappings (7) into itself, thus preserving Eq.(8). Hence

$$d\psi(p^*)/d\varphi^* = \psi'(\rho^*) f(\rho^*) = f'(\rho^*)\psi(\rho^*) \tag{9}$$

The solutions of Eq.(9) are determined by formula $\psi(\rho^*) = Cf(\rho^*)$. After appropriate normalization, taking into account expansion (3), we obtain $A_k = a_1^{(k)}(0)$.

Thus, for each equation of phase trajectories a unique method of obtaining its special "normal" form (8) has been indicated. System (1) admits an analytic symmetry group only when the normal form (8) is convergent.

Briuno's theorem III on divergence /2/ was first proved by him (**) for real systems (1) (of very particular form only). According to Theorem III not every system E(1) can be normalized by a convergent transformation.

A criterion of analytical normalization of systems (1) has been formulated above in terms of analytic properties of phase trajectories. The problem of formulating the criterion in terms of right-hand sides of R and Ω of Eqs.(1) in explicit form remains so far unresolved.

REFERENCES

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*) A.D. Briuno, On noncanonical invariants of Hamiltonian systems. Preprint No.4, Inst. Prikl. Matem., Akad. Nauk SSSR, 1979.

**) A.D. Briuno, Divergence of the real normalizing transformation. Preprint No.62, Inst. Prikl. Matem, Akad. Nauk SSSR. Moscow, 1979.

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